

Qualitative investigation of the solutions to differential equations

(Application of the skew-symmetric differential forms)

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The presented method of investigating the solutions to differential equations is not new. Such an approach was developed by Cartan [1] in his analysis of the integrability of differential equations. Here this approach is outlined to demonstrate the role of skew-symmetric differential forms.

The role of skew-symmetric differential forms in a qualitative investigation of the solutions to differential equations is conditioned by the fact that the mathematical apparatus of these forms enables one to determine the conditions of consistency for various elements of differential equations or for the system of differential equations. This enables one, for example, to define the consistency of the partial derivatives in the partial differential equations, the consistency of the differential equations in the system of differential equations, the conjugacy of the function derivatives and of the initial data derivatives in ordinary differential equations and so on. The functional properties of the solutions to differential equations are just depend on whether or not the conjugacy conditions are satisfied.

Specific features of the solutions to differential equations

The basic idea of the qualitative investigation of the solutions to differential equations can be clarified by the example of the first-order partial differential equation.

Let

$$F(x^i, u, p_i) = 0, \quad p_i = \partial u / \partial x^i \quad (1)$$

be the partial differential equation of the first order. Let us consider the functional relation

$$du = \theta \quad (2)$$

where $\theta = p_i dx^i$ (the summation over repeated indices is implied). Here $\theta = p_i dx^i$ is the differential form of the first degree.

The specific feature of functional relation (2) is that in the general case this relation turns out to be nonidentical.

The left-hand side of this relation involves a differential, and the right-hand side includes the differential form $\theta = p_i dx^i$. For this relation to be identical, the differential form $\theta = p_i dx^i$ must be a differential as well (like the left-hand side of relation (2)), that is, it has to be a closed exterior differential form. To do this it requires the commutator $K_{ij} = \partial p_j / \partial x^i - \partial p_i / \partial x^j$ of the differential form θ has to vanish.

In the general case, from equation (1) it does not follow (explicitly) that the derivatives $p_i = \partial u / \partial x^i$, which obey to the equation (and given boundary or initial conditions of the problem), make up a differential. Without any supplementary conditions the commutator of the differential form θ defined as $K_{ij} = \partial p_j / \partial x^i - \partial p_i / \partial x^j$ is not equal to zero. The form $\theta = p_i dx^i$ occurs to be unclosed and is not a differential like the left-hand side of relation (2). The functional relation (2) appears to be nonidentical: the left-hand side of this relation is a differential, but the right-hand side is not a differential.

The nonidentity of functional relation (2) points to a fact that without additional conditions derivatives of the initial equation do not make up a differential. This means that the corresponding solution to the differential equation u will not be a function of x^i . It will depend on the commutator of the form θ , that is, it will be a functional.

To obtain the solution that is a function (i.e., derivatives of this solution form a differential), it is necessary to add the closure condition for the form $\theta = p_i dx^i$ and for the dual form (in the present case the functional F plays a role of the form dual to θ) [1]:

$$\begin{cases} dF(x^i, u, p_i) = 0 \\ d(p_i dx^i) = 0 \end{cases} \quad (3)$$

If we expand the differentials, we get a set of homogeneous equations with respect to dx^i and dp_i (in the $2n$ -dimensional space – initial and tangential):

$$\begin{cases} \left(\frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} p_i \right) dx^i + \frac{\partial F}{\partial p_i} dp_i = 0 \\ dp_i dx^i - dx^i dp_i = 0 \end{cases} \quad (4)$$

The solvability conditions for this system (vanishing of the determinant composed of coefficients at dx^i, dp_i) have the form:

$$\frac{dx^i}{\partial F / \partial p_i} = \frac{-dp_i}{\partial F / \partial x^i + p_i \partial F / \partial u} \quad (5)$$

These conditions determine an integrating direction, namely, a pseudostructure, on which the form $\theta = p_i dx^i$ occurs to be closed one, i.e. it becomes a differential, and from relation (2) the identical relation is produced. If conditions (5), that may be called the integrability conditions, are satisfied, the derivatives constitute a differential $\delta u = p_i dx^i = du$ (on the pseudostructure), and the solution becomes a function. Just such solutions, namely, functions on the pseudostructures formed by the integrating directions, are the so-called generalized solutions [2]. The derivatives of the generalized solution constitute the exterior form, which is closed on the pseudostructure.

(If conditions (5) are not satisfied, that is, the derivatives do not form a differential, the solution that corresponds to such derivatives will depend on the

differential form commutator constructed of derivatives. That means that the solution is a functional rather than a function.)

Since the functions that are the generalized solutions are defined only on the pseudostructures, they have discontinuities in derivatives in the directions being transverse to the pseudostructures. The order of derivatives with discontinuities is equal to the exterior form degree. If the form of zero degree is involved in the functional relation, the function itself, being a generalized solution, will have discontinuities.

If we find the characteristics of equation (1), it appears that conditions (5) are the equations for characteristics [3]. That is, the characteristics are examples of the pseudostructures on which derivatives of the differential equation constitute the closed forms and the solutions turn out to be functions (generalized solutions). (The characteristic manifolds of equation (1) are the pseudostructures π on which the form $\theta = p_i dx^i$ becomes a closed form: $\theta_\pi = du_\pi$).

Here it is worth noting that coordinates of the equations for characteristics are not identical to independent coordinates of initial space on which equation (1) is defined. The transition from initial space to the characteristic manifold appears to be a *degenerate* transformation, namely, the determinant of the system of equations (4) becomes zero. The derivatives of equation (1) are transformed from the tangent space to the cotangent one. The transition from the tangent space, where the commutator of the form θ is nonzero (the form is unclosed, the derivatives do not form a differential), to the characteristic manifold, namely, the cotangent space, where the commutator becomes equal to zero (the closed exterior form is formed, i.e. the derivatives make up a differential), is the example of the degenerate transformation.

Skew-symmetric differential forms, which, in contrast to exterior skew-symmetric differential forms, are defined on manifolds with unclosed differential forms, were considered in the author's work [4]. Such skew-symmetric differential forms, which were named evolutionary differential forms (since they possess the evolutionary properties) cannot be closed forms. They emerge while describing real processes by differential equations. The skew-symmetric differential form $\theta = p_i dx^i$, which enters into functional relation (2), is the example of evolutionary skew-symmetric differential forms. Since the evolutionary skew-symmetric differential form is unclosed, the relation with such differential form turns out to be nonidentical. The properties of such nonidentical relation just specify functional properties of the solutions to differential equations [4,5].

The partial differential equation of the first order has been analyzed, and the functional relation with the form of the first degree analogous to the evolutionary form has been considered.

Similar functional properties have the solutions to all differential equations. And, if the order of the differential equation is k , the functional relation with the k -degree form corresponds to this equation. For ordinary differential equations the commutator is produced at the expense of the conjugacy of derivatives of the functions desired and those of the initial data (the dependence of the solution

on the initial data is governed by the commutator).

In a similar manner one can also investigate the solutions to a system of partial differential equations and the solutions to ordinary differential equations (for which the nonconjugacy of desired functions and initial conditions is examined).

It can be shown that the solutions to equations of mathematical physics, on which no additional external conditions are imposed, are functionals. The solutions prove to be exact only under realization of the additional requirements, namely, the conditions of degenerate transformations: vanishing determinants, Jacobians and so on, that define the integral surfaces. The characteristic manifolds, the envelopes of characteristics, singular points, potentials of simple and double layers, residues and others are the examples of such surfaces.

Here the mention should be made of the generalized Cauchy problem when the initial conditions are given on some surface. The so called “unique” solution to the Cauchy problem, when the output derivatives can be determined (that is, when the determinant built of the expressions at these derivatives is nonzero), is a functional since the derivatives obtained in such a way prove to be nonconjugated, that is, their mixed derivatives form a commutator with nonzero value, and the solution depends on this commutator.

The dependence of the solution on the commutator can lead to instability of the solution. Equations that do not provided with the integrability conditions (the conditions such as, for example, the characteristics, singular points, integrating factors and others) may have the unstable solutions. Unstable solutions appear in the case when the additional conditions are not realized and no exact solutions (their derivatives form a differential) are formed. Thus, the solutions to the equations of the elliptic type can be unstable.

Investigation of nonidentical functional relations lies at the basis of the qualitative theory of differential equations. It is well known that the qualitative theory of differential equations is based on the analysis of unstable solutions and integrability conditions. From the functional relation it follows that the dependence of the solution on the commutator leads to instability, and the closure conditions of the forms constructed by derivatives are the integrability conditions. One can see that the problem of unstable solutions and integrability conditions appears, in fact, to be reduced to the question of under what conditions the identical relation for the closed form is produced from the nonidentical relation that corresponds to the relevant differential equation (the relation such as (2)), the identical relation for the closed form is produced. In other words, whether or not the solutions are functionals? This is to the same question that the analysis of the correctness of setting the problems of mathematical physics is reduced.

Here the following should be emphasized. When the degenerate transformation from the initial nonidentical functional relation is performed, an integrable identical relation is obtained. As the result of integrating, one obtains a relation that contains exterior forms of less by one degree and which once again proves to be (in the general case without additional conditions) nonidentical.

By integrating the functional relations obtained sequentially (it is possible only under realization of the degenerate transformations) from the initial functional relation of degree k one can obtain $(k + 1)$ functional relations each involving exterior forms of one of degrees: $k, k - 1, \dots, 0$. In particular, for the first-order partial differential equation it is also necessary to analyze the functional relation of zero degree.

Thus, application of the skew-symmetric differential forms allows one to reveal the functional properties of the solutions to differential equations.

Analysis of field equations

Field theory is based on the conservation laws. The conservation laws are described by the closure conditions of the exterior differential forms [6]. It is evident that the solutions to the equations of field theory describing physical fields can be only generalized solutions, which correspond to closed exterior differential forms.

The generalized solutions (i.e. solutions whose derivatives form a differential, namely, the closed form), can have a differential equation, which is subject to the additional conditions.

Let us consider what equations are obtained in this case.

Return to equation (1).

Assume that the solution does not explicitly depend on u and is resolved with respect to some variable, for example t , that is, it has the form of

$$\frac{\partial u}{\partial t} + E(t, x^j, p_j) = 0, \quad p_j = \frac{\partial u}{\partial x^j} \quad (6)$$

Then integrability conditions (5) (the closure conditions of the differential form $\theta = p_i dx^i$ and the corresponding dual form) can be written as (in this case $\partial F / \partial p_1 = 1$)

$$\frac{dx^j}{dt} = \frac{\partial E}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial E}{\partial x^j} \quad (7)$$

These are the characteristic relations for equation (6). As it is well known, the canonical relations have just such a form.

As a result we conclude that the canonical relations are the characteristics of equation (6) and the integrability conditions for this equation.

The canonical relations obtained from the closure condition of the differential form $\theta = p_i dx^i$ and the corresponding dual form, are the examples of the identical relation of the theory of exterior differential forms.

Equation (6) provided with the supplementary conditions, namely, the canonical relations (7), is called the Hamilton-Jacobi equation [3]. In other words, the equation whose derivatives obey the canonical relation is referred to as the Hamilton-Jacobi equation. The derivatives of this equation form the differential, i.e. the closed exterior differential form: $\delta u = (\partial u / \partial t) dt + p_j dx^j = -E dt + p_j dx^j = du$.

The equations of field theory belong to this type.

$$\frac{\partial s}{\partial t} + H\left(t, q_j, \frac{\partial s}{\partial q_j}\right) = 0, \quad \frac{\partial s}{\partial q_j} = p_j \quad (8)$$

where s is the field function for the action functional $S = \int L dt$. Here L is the Lagrange function, H is the Hamilton function: $H(t, q_j, p_j) = p_j \dot{q}_j - L$, $p_j = \partial L / \partial \dot{q}_j$. The closed form $ds = -H dt + p_j dq_j$ (the Poincare invariant) corresponds to equation (8).

The coordinates q_j, p_j in equation (A.8) are conjugated ones. They obey the canonical relations.

In quantum mechanics (where to the coordinates q_j, p_j the operators are assigned) the Schrödinger equation serves as an analog to equation (8), and the Heisenberg equation serves as an analog to the relevant equation for the canonical relation integral. Whereas the closed exterior differential form of zero degree (the analog to the Poincare invariant) corresponds to the Schrödinger equation, the closed dual form corresponds to the Heisenberg equation.

A peculiarity of the degenerate transformation can be considered by the example of the field equation. The transition from the unclosed differential form (which is included into the functional relation) to the closed form is the degenerate transformation. Under degenerate transformation the transition from the initial manifold (on which the differential equation is defined) to the characteristic (integral) manifold goes on.

Here the degenerate transformation is a transition from the Lagrange function to the Hamilton function. The equation for the Lagrange function, that is the Euler variational equation, was obtained from the condition $\delta S = 0$, where S is the action functional. In the real case, when forces are nonpotential or couplings are nonholonomic, the quantity δS is not a closed form, that is, $d\delta S \neq 0$. But the Hamilton function is obtained from the condition $d\delta S = 0$ which is the closure condition for the form δS . The transition from the Lagrange function L to the Hamilton function H (the transition from variables q_j, \dot{q}_j to variables $q_j, p_j = \partial L / \partial \dot{q}_j$) is a transition from the tangent space, where the form is unclosed, to the cotangent space with a closed form. One can see that this transition is a degenerate one.

The invariant field theories used only nondegenerate transformations that conserve a differential. By the example of the canonical relations it is possible to show that nondegenerate and degenerate transformations are connected. The canonical relations in the invariant field theory correspond to nondegenerate tangent transformations. At the same time, the canonical relations coincide with the characteristic relation for equation (8), which the degenerate transformations correspond to. The degenerate transformation is a transition from the tangent space (q_j, \dot{q}_j) to the cotangent (characteristic) manifold (q_j, p_j) . On the other hand, the nondegenerate transformation is a transition from one characteristic manifold (q_j, p_j) to another characteristic manifold (Q_j, P_j) . {The

formula of canonical transformation can be written as $p_j dq_j = P_j dQ_j + dW$, where W is the generating function}.

It may be easily shown that such a property of duality is also a specific feature of transformations such as tangent, gradient, contact, gauge, conform mapping, and others.

Thus, the application of mathematical apparatus of the skew-symmetric differential forms to qualitative investigation of the solutions to differential equations enables one to understand the specific features of the solutions to differential equations. Such investigation is of interest in the analysis of the equations of mathematical physics.

References

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